From late 19^{th}, early 20^{th} century:

light: wave and particle (photon)

matter: particle and wave (de Broglie)

so, if matter behaves like a wave, what is the wave function?

ex: periodic traveling water waves:

$$\Psi(x,t) = A\sin(kx - \omega t)$$

Ψ is the amplitude of the displacement of water at position x, time t

for a particle, what does $\Psi(x,t)$ mean?

$\Rightarrow$ this is one of the fundamental things about <u>Quantum Mechanics</u>

current interpretation:

e.g. $\Psi(x,t)$ for an electron = "amplitude" for finding the electron at x, t

Ψ can be a complex number: has real and imaginary part

Review of complex numbers

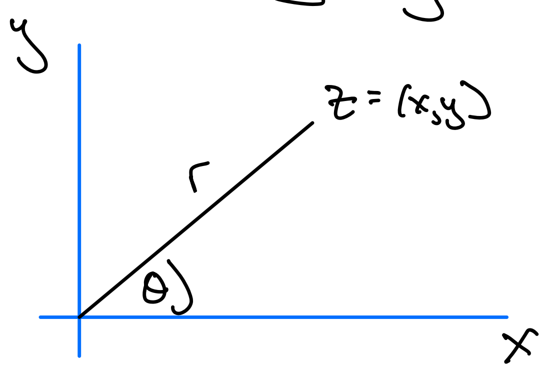
let z be complex number

let $i = \sqrt{-1}$ so $i^2 = -1$

can write $z = x + iy$

x = "real part of z" $= \mathrm{Re}(z)$

y = "imaginary part of z" $= \mathrm{Im}(z)$



can write $x = r\cos\theta$

$y = r\sin\theta$

so $z = r\cos\theta + i\, r\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$\sin\theta$ & $\cos\theta$ can be expanded around $\theta = 0$ using the usual Taylor series:

$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \dots$

$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$

so $\cos\theta + i\sin\theta = 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \dots$

now use $i^2 = -1$

so $-\frac{\theta^2}{2!} = +\frac{(i\theta)^2}{2!}$

$-i\frac{\theta^3}{3!} = i \cdot i^2 \frac{\theta^3}{3!} = \frac{(i\theta)^3}{3!}$

$\frac{\theta^4}{4!} = (i^2)^2 \frac{\theta^4}{4!} = \frac{(i\theta)^4}{4!}$

and so on

so can write

$$\cos\theta + i\sin\theta = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots$$

now expand e^x for small x using Taylor expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

so $\cos\theta + i\sin\theta = e^{i\theta}$!

then can write $z = re^{i\theta} = x + iy$

where $x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2+y^2}$

$\tan\theta = y/x$

"complex conjugate" of z defined as

$z^* = re^{-i\theta} = x - iy$

$z = re^{+i\theta} = x + iy$

$|z|$ is "absolute value" of complex number z and $z = r = \sqrt{x^2+y^2}$

try $z \cdot z^* = (x+iy)(x-iy)$

$= x^2 + iyx - iyx - (iy)^2$

$= x^2 + y^2$

so $|z| = \sqrt{zz^*} = \sqrt{x^2+y^2} = \sqrt{Re(z)^2 + Im(z)^2}$

complex numbers are very useful for many engineering applications (like AC circuits)

back to QM:

$\Psi(x,t)$ is complex, amplitude for finding a particle at position x, time t

what does amplitude mean?

current interpretation:

$|\Psi(x,t)|^2$ = probability of finding particle at location x, time t

but probability is a tricky thing - probability of finding particle <u>exactly</u> at x,t gets smaller as you make x, t more precise

so define probability of finding particle between location x and x + dx is:

$$P(x, x+dx) = |\Psi(x,t)|^2 dx$$

so $|\Psi(x,t)|^2$ is probability per length

or "probability density"

$$\text{so } P(x, x+\Delta x) = \int_x^{x+\Delta x} |\Psi(x,t)|^2 dx$$

which means

$$P(x=-\infty, x=+\infty) = \int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1$$

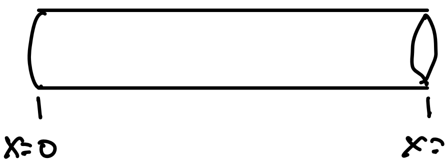
because particle has to be somewhere!

ex: particle is equally likely to be anywhere in a tube of length L

$\Rightarrow$ if particle is equally likely to be at any location in tube, then there is no time dependence at each location

$$\text{so } \Psi(x,t) = \psi(x) = C \quad \text{for } 0 \le x \le L$$
$$= 0 \quad \text{for } x < 0 \text{ or } x > L$$

where C is a constant



probability normalization requires:

$$\int_{-\infty}^{\infty} \Psi(x,t)^* \Psi(x,t)\, dx = \int_{-\infty}^{\infty} \psi(x)^* \psi(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} |\psi|^2 dx = \underbrace{\int_{-\infty}^{0} |\psi|^2 dx}_{\psi=0} + \underbrace{\int_{0}^{L} |\psi|^2 dx}_{\psi=C} + \underbrace{\int_{L}^{+\infty} |\psi|^2 dx}_{\psi=0}$$

$$= \int_0^L C^2\, dx = C^2 L = 1$$

so $\quad C = 1/\sqrt{L}$

what is prob to find particle in 1st half of tube?

$$P(0,L) = \int_0^{L/2} |\psi(x)|^2 dx = \frac{1}{L}\int_0^{L/2} dx = \frac{1}{L}\cdot\frac{L}{2} = \frac{1}{2} \checkmark$$

ex: free particle moving w/ $v = \frac{\omega}{k}$ $\omega = 2\pi f$, $k = \frac{2\pi}{\lambda}$

$$\psi(x,t) = Ae^{i(kx-\omega t)} = A\cos(kx-\omega t) + Ai\sin(kx-\omega t)$$

this is the most general wave equation for a free particle

what is probability of particle being between x_1 and x_2 locations?

$$P(x_1,x_2) = \int_{x_1}^{x_2} |\psi|^2 dx = \int_{x_1}^{x_2} \overset{\psi}{Ae^{i(kx-\omega t)}} \, \overset{\psi^*}{Ae^{-i(kx-\omega t)}} dx$$

$$= A^2 \int_{x_1}^{x_2} dx = A^2 (x_2 - x_1) = A^2 \Delta x$$

but shouldn't the particle have a definite location?

QM says: 1. before you measure it, a particle does <u>not</u> have a definite location

2. but it can have a definite probability

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Properties of particles

ex: what is average position of a particle described by wave function $\Psi(x,t)$?

classical statistics: average age of all students

$$\overline{age} = \frac{\sum_{i=1}^{N} age_i}{N} \qquad \frac{\text{sum all ages}}{\#\text{students}}$$

rewrite: $\overline{age} = \sum_{i=1}^{N} \frac{1}{N} * age_i$

sum each age weighted by $\frac{1}{N}$

"weight" is same for all students

can say that weight $\frac{1}{N}$ is probability for each student: $P(age_i) = \frac{1}{N}$

so average is $\overline{age} = \sum_{i=1}^{N} P(age_i) \cdot age_i$

so for any average x: $\bar{x} = \sum x \cdot P(x)$

in QM we use amplitude $\Psi(x)$ and write something similar:

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$$\bar{x} = \int_{-\infty}^{\infty} \Psi^*(x) \cdot x \cdot \Psi(x)\, dx$$

ex: particle is in a box of length L

wave function $\Psi(x,t)$ is not a travelling wave!

assume: • $\Psi(x,t) = \psi(x)\,\phi(t)$

$\Rightarrow$ $\psi(x)$ is function of position

$\Rightarrow$ at any x $\phi(t)$ is how amplitude changes

ex:

for $\psi(x)$ try $\psi(x) = A\sin(kx)$ $\quad 0 \le x \le L$

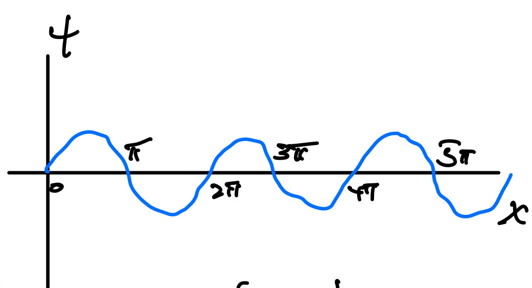
$= 0 \quad x < 0,\ x > L$ outside boundary

if $\psi(x<0, x>L) = 0$ then by continuity, want

$\psi(x=0) = \psi(x=L) = 0$ at boundaries

$\Rightarrow$ $\psi(0) = A\sin(0) = 0$ ✓

$\psi(L) = A\sin(kL) = 0$



so $\psi(L) = 0$ when $kL = n\pi \quad n = 0, 1, 2 \ldots$

or $k_n = n \cdot \frac{\pi}{L}$ $\quad k_n$ is quantized

but then we always get discrete values of wavelengths when we constrain a wave to have boundaries

e.g. violin, guitar, ...

normalize: $\displaystyle\int_{-\infty}^{\infty} \psi^*(x)\,\psi(x)\,dx = 1$

$$\int_{-\infty}^{\infty}|\psi(x)|^2dx = \int_{-\infty}^{0}|\psi(x)|^2dx + \int_{0}^{L}|\psi(x)|^2dx + \int_{L}^{\infty}|\psi(x)|^2dx$$

$$= \int_0^L|\psi(x)|^2dx = A^2\int_0^L \sin^2 kx\,dx$$

$$= A^2\int_0^L \tfrac{1}{2}(1-\cos 2kx)\,dx$$

$$= A^2\left[\frac{L}{2} - \frac{\sin 2kx}{2k}\Big|_0^L\right]$$

$$\sin 2kL = \sin\left(2\cdot\frac{n\pi}{L}\cdot L\right) = 0$$

so $\displaystyle\int_0^L|\psi(x)|^2dx = \frac{A^2L}{2} = 1$

means $A = \sqrt{\dfrac{2}{L}}$

$\psi(x) = \sqrt{\dfrac{2}{L}}\sin(k_nx)$ $k_n = \dfrac{n\pi}{L}$ $n =$

deBroglie: $p = \dfrac{h}{\lambda} = \dfrac{h}{2\pi}\cdot\dfrac{2\pi}{\lambda} = \hbar k$

so if k is quantized so is p!

$p_n = \hbar k_n = n\cdot\dfrac{\hbar\pi}{L}$ $n = 0, 1, 2, \ldots$

consider $n=0$: then $\psi(x) = \sqrt{\dfrac{2}{L}}\sin(k_0x) = 0$

but if $\psi(x)=0$ inside the box, then no particle anywhere!

$\Rightarrow$ this tells you

1. $n=0$ not allowed
2. $p_n = n\frac{\hbar\pi}{L}$ $n=1,2,3\ldots$ so p_0 does not exist!

$\Rightarrow$ particle can not be sitting still with no velocity

$\Rightarrow$ QM says even if you cool the particle to absolute 0 temp $(T=0)$ it has to have a non-zero momentum and therefore a non-zero energy!

non relativistic energy $E = KE = \frac{p^2}{2m}$

$$\text{so } E_n = \frac{p_n^2}{2m} = \left(n\frac{\hbar\pi}{L}\right)^2 \frac{1}{2m} = n^2 \frac{(\hbar\pi)^2}{2mL}$$

energy is also quantized!

now calculate average position of particle in box

$$\bar{x} = \int_0^L \psi(x)^* x\, \psi(x)\,dx \qquad \psi(x) = \sqrt{\frac{2}{L}}\sin(k_n x)$$

so ψ is real so $\psi^* = \psi$

then $\bar{x} = \int_0^L |\psi(x)|^2 x\,dx = \int_0^L \frac{2}{L} x \sin^2 x\,dx$

$$= \frac{2}{L}\left[\frac{x^2}{4} - \frac{x\sin(2kx)}{4k} - \frac{\cos(2kx)}{8k^2}\right]_0^L$$

$\sin 2kL = \sin\left(2\cdot\frac{n\pi}{L}\cdot L\right) = 0$

$\sin(0) = 0$

$\cos(2kL) = \cos\left(2\cdot\frac{n\pi}{L}\cdot L\right) = \cos(2\pi\cdot n) = 1$

$\cos(0) = 1$

so only 1st term survives

$$\bar{x} = \frac{2}{L}\frac{L^2}{4} = \frac{L}{2} \quad \text{as expected}$$

but what does "average position" mean?

$\Rightarrow$ averaged over <u>many</u> measurements!

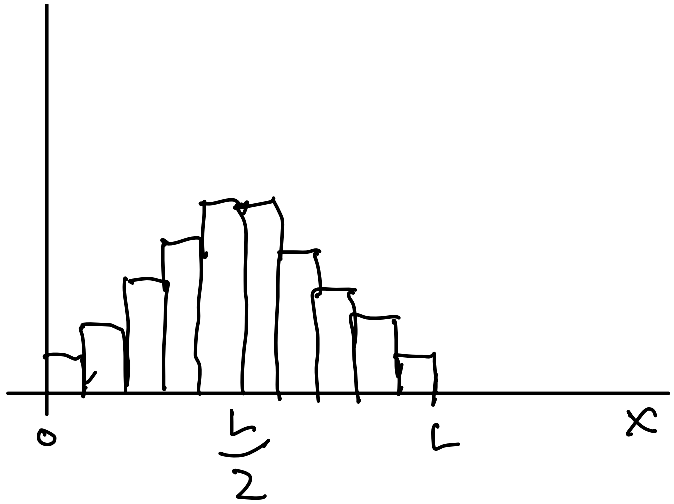
so if you perform many measurements and count ~~#~~ times you measure x between 0 - 0.1L

0.1L - 0.2L
0.2L - 0.3L
⋮
0.9L - 1.0L

this is a "histogram"

plot will show

average at $\frac{L}{2}$



<u>Wave equation</u>

in E & M, for traveling EM wave we have wave equation

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 E}{\partial t^2} = 0$$

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E is a wave that satisfies that wave equation

for quantum mechanics:

if $\Psi(x,t)$ is a wave function, there must be a wave equation

for a free particle of mass m:

$$\Psi(x,t) = A e^{i(kx-\omega t)} = A\cos(kx-\omega t) + i\sin(kx-\omega t)]$$

with the deBroglie relations

$$p = \frac{h}{\lambda} = \frac{h}{2\pi}\cdot\frac{2\pi}{\lambda} = \hbar k$$

$$E = hf = \frac{h}{2\pi}\cdot 2\pi f = \hbar\omega$$

so can write

$$\Psi(x,t) = A e^{i\left(\frac{p}{\hbar}x - \frac{E}{\hbar}t\right)} = Ae^{i\left(\frac{px - Et}{\hbar}\right)}$$

if you take $\frac{\partial \Psi}{\partial t}$ you get

$$\frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial t} Ae^{i\left(\frac{px-Et}{\hbar}\right)}$$

$$= \left(-\frac{iE}{\hbar}\right) Ae^{i\left(\frac{px-Et}{\hbar}\right)}$$

$$= -\frac{i}{\hbar} E \Psi$$

so write $i\hbar \frac{\partial \Psi}{\partial t} = i\hbar\left(\frac{-i}{\hbar}\right) E\Psi = E\Psi$

we can interpret $i\hbar \frac{\partial}{\partial t}$ as an "operator" $\hat{E}$

where $\underset{\text{operator}}{\hat{E}}\Psi = \underset{\text{energy value}}{E}\Psi$

also try $\frac{\partial}{\partial x}\Psi = \frac{\partial}{\partial x} Ae^{i\left(\frac{px-Et}{\hbar}\right)}$

$$= \frac{ip}{\hbar} Ae^{i\left(\frac{px-Et}{\hbar}\right)} = \frac{ip\Psi}{\hbar}$$

so momentum operator $\hat{p} = -i\hbar \frac{\partial}{\partial x}$

and $\hat{p}\psi = p\psi$

think of these operators as "projecting" quantities from wave functions

Quantum theory: can use momentum and energy operators to find average values of momentum & energy

ex: for particle in a box, no time dependence of the box

so $\psi(x) = \sqrt{\frac{2}{L}} \sin(kx)$ $\quad k = \frac{n\pi}{L}$ as above

and average x is $\bar{x} = \frac{L}{2}$

what is average momentum?
(AKA "expectation value" of momentum)

$$\bar{p} = \int_0^L \psi^*(x)\, \hat{p}\, \psi(x)\, dx$$

$$= \int_0^L \sqrt{\frac{2}{L}} \sin(kx) * -i\hbar \frac{\partial}{\partial x} \sqrt{\frac{2}{L}} \sin(kx)\, dx$$

$$= -2i\frac{\hbar k}{L} \int_0^L \sin(kx)\cos(kx)\, dx = 0$$

because particle spends equal time going in both directions inside box!

what is average energy?

for $\bar{E}$ use energy operator $\hat{E} = \frac{\hat{p}^2}{2m} = \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x}\right)\left(-i\hbar\frac{\partial}{\partial x}\right)$

$$= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

$$\bar{E} = \int_0^L \psi^* \hat{E} \psi \, dx$$

$$= \int_0^L \sqrt{\frac{2}{L}} \sin(kx) \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\sqrt{\frac{2}{L}} \sin(kx)\right) dx$$

$$\frac{\partial^2}{\partial x^2} \sin(kx) = k\frac{\partial}{\partial x}\cos(kx) = -k^2 \sin(kx)$$

so $$\bar{E} = \frac{2}{L}\int_0^L \sin^2(kx) \cdot \frac{\hbar^2}{2m}k^2 \, dx$$

$$= \frac{\hbar^2 k^2}{mL}\int_0^L \sin^2(kx) dx$$

$$= \frac{\hbar^2 k^2}{mL}\int_0^L \left(\frac{1-\cos(2kx)}{2}\right) dx$$

$$= \frac{\hbar^2 k^2}{mL} \cdot \frac{L}{2} = \frac{\hbar^2}{2m} \cdot \left(\frac{n\pi}{L}\right)^2$$

so $\bar{E}_n = n^2 \frac{\hbar^2 \pi^2}{2mL^2}$ is also quantized

so average energy is just the energy for the particle in the n'th state

Schrodinger equation

non-relativistically, we have

$$E = KE = \frac{1}{2}mv^2 = \frac{1}{2}\frac{m^2v^2}{m} = \frac{p^2}{2m}$$

so $\frac{p^2}{2m} = E$

so why not $\frac{p^2}{2m}\Psi = E\Psi$! $(\Psi(x,t))$

now write out the operators:

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

$$\hat{p}^2 = -\hbar^2\frac{\partial^2}{\partial x^2}$$

$$\hat{E} = i\hbar\frac{\partial}{\partial t}$$

turn operator equation into differential equation for a free particle:

$$\hat{KE}\Psi = \frac{\hat{p}^2}{2m}\Psi \Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$$

$$\hat{E}\Psi \Rightarrow i\hbar\frac{\partial\Psi}{\partial t}$$

so energy equation $\hat{KE}\Psi = \hat{E}\Psi$ gives us

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

what about a particle <u>not</u> free?

e.g. electron in H atom

$\Rightarrow$ not free is because of potential energy!

$\Rightarrow$ PE's almost always do <u>not</u> have any time dependence

let $U(x)$ be potential energy as a function of position

then imagin PE operator $\hat{PE} = \hat{U}(x)$

classically: $E = KE + PE$

so extend free particle diff eqn above:

$$\underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U(x)\Psi}_{\text{total energy operator } (\hat{KE} + \hat{U})\Psi} = i\hbar\frac{\partial\Psi}{\partial t} \Leftarrow \text{Schrodinger equation}$$

note: if $U(x)$ only and not a function of time

then $\Psi(x,t) = \psi(x)e^{-iEt/\hbar} = \psi(x)e^{-i\omega t}$

is a solution because

$$i\hbar \frac{\partial \Psi}{\partial t} = i\hbar\, \psi(x) \frac{\partial}{\partial t} e^{-i\omega t}$$
$$= i\hbar\, \psi(x)(-i\omega) e^{-i\omega t}$$
$$= \hbar\omega\, \psi(x) e^{-i\omega t} = \hbar\omega\, \Psi(x,t)$$

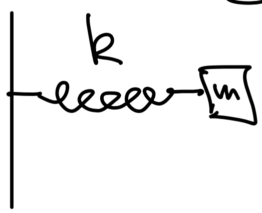
but $\hbar\omega = E$ from deBroglie

so operator eqn is

$$\widehat{KE}\, \Psi(x,t) + \hat{U}(x)\Psi(x,t) = E\Psi(x,t) \checkmark$$

Harmonic oscillator

like a spring and mass:



k is "spring constant" from Hooke's Law $F = -kx$

potential energy is $PE = \frac{1}{2}kx^2$

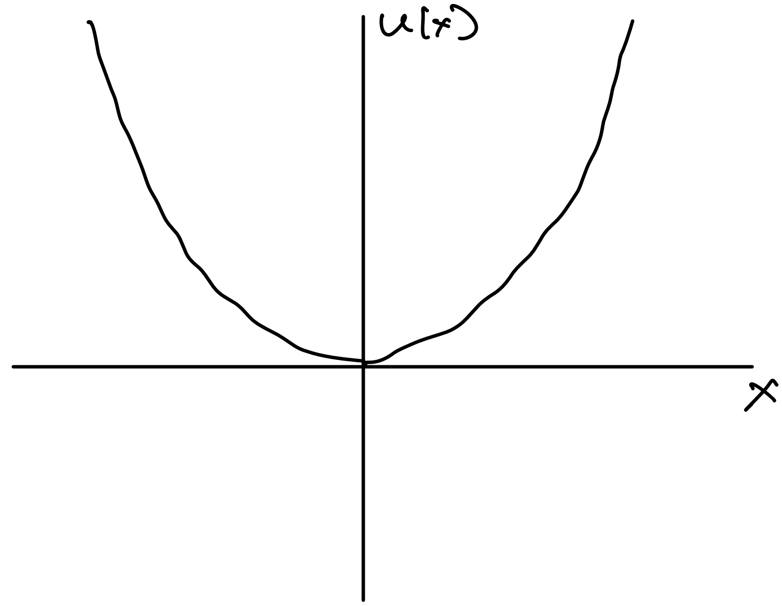
so for QM oscillator $U(x) = \frac{1}{2}kx^2$

and $\omega = \sqrt{\frac{k}{m}}$ so $k = m\omega^2$

$$U(x) = \frac{1}{2}m\omega^2 x^2$$

the potential energy looks like this:

potential



Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial t^2}+\frac{1}{2}m\omega^2x^2\psi(x)=E\,\psi(x)$$

this is a difficult differential eqn

answer: $\psi_n(x) = N_n\, e^{-\frac{\beta^2 x^2}{2}} H_n(\beta x)$

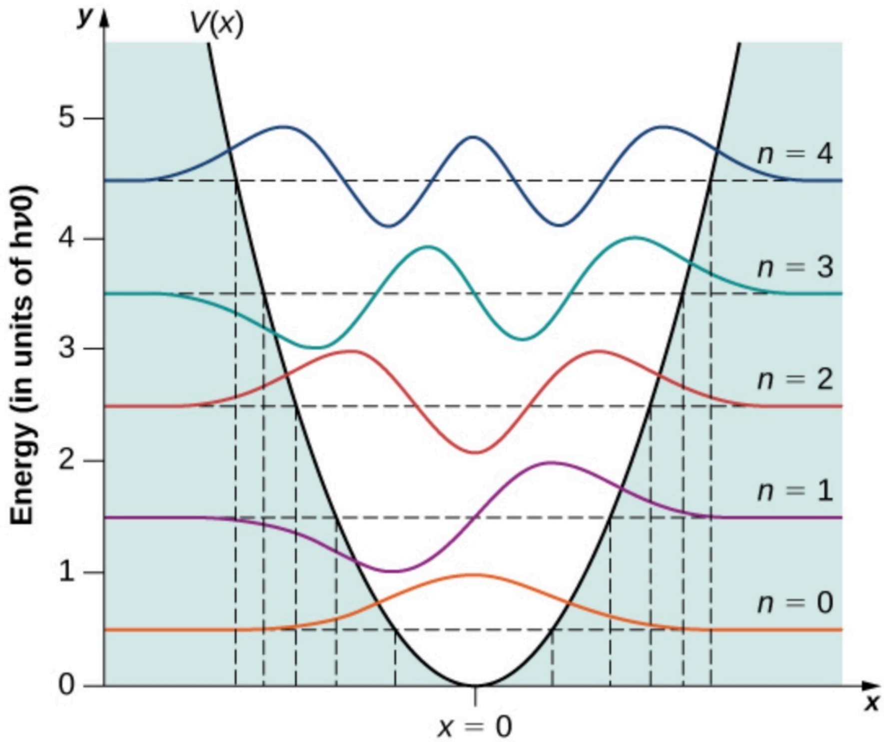
wave function for each energy level n

normalization

$\beta = \sqrt{\frac{m\omega}{\hbar}}$

"Hermite polynomial"

wave function looks like this:



energy is found to be

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega \qquad n = 0, 1, 2, \ldots$$

ground state: $n=0$, $E_0 = \frac{1}{2}\hbar\omega$

1st excited state: $n=1$, $E_1 = \frac{3}{2}\hbar\omega$

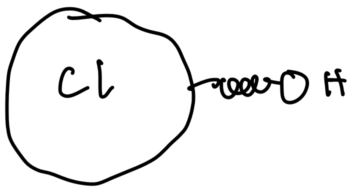
⋮

energy levels separated by

$$\Delta E = E_{n+1} - E_n$$

$$= \left(n+1+\frac{1}{2}\right)\hbar\omega - \left(n+\frac{1}{2}\right)\hbar\omega = \hbar\omega$$

can use this to model quantised energy levels of molecules like hydrogen chloride HCl

- Cl molecule mass is 35x more than H
- 

⇒ observed oscillation lowest frequency measured to be 88.8 THz ($THz = 10^{12}$ Hz)

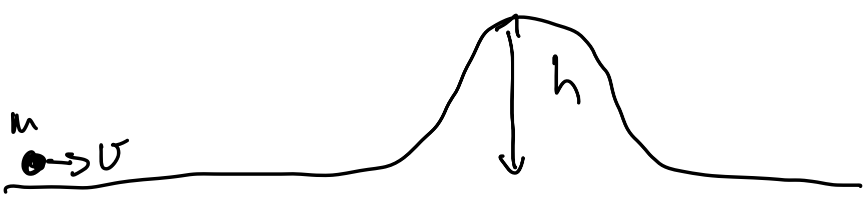
⇒ $\Delta E = hf = 6.63 \times 10^{-34}$ J·s $\times 88.8 \times 10^{12} \frac{1}{s} \times \frac{1\,eV}{1.6 \times 10^{-19} J}$

$= 0.37$ eV

Quantum Tunneling

classically: $E_{tot} = KE + PE$

ex: ball and hill



ball has $KE = \frac{1}{2} m v^2$

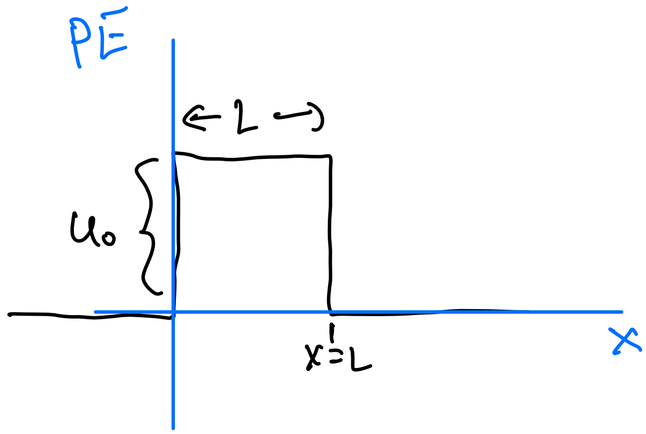
PE at top of hill $PE = mgh$

- as ball goes up the hill, KE is converted to PE

• ball needs KE $\geq$ PE to get to the top and go over to other side

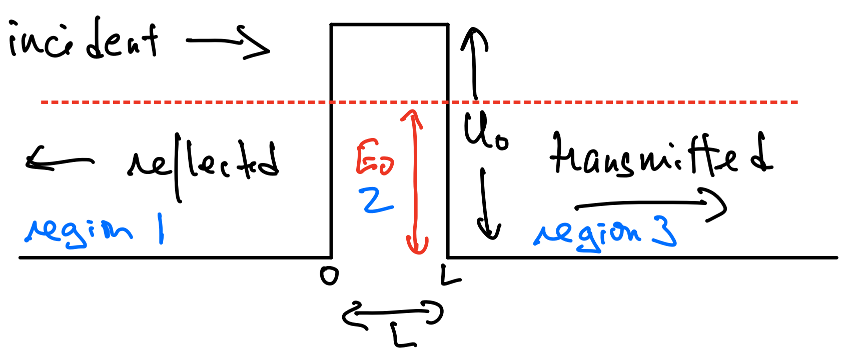
in quantum world:

let $U(x) = \begin{cases} 0 & x<0 \\ U_0 & 0 \leq x \leq L \\ 0 & x > L \end{cases}$ $\quad U_0$ constant



$U(x)$ acts like a (energy) potential barrier

$\Rightarrow$ a particle is a wave coming in from $x = -\infty$ and hits barrier. Particle has energy $E_0 < U_0$



this situation is described by Schrodinger equation

$\Rightarrow$ suppose situation has no time dependence

$\Psi(x,t) = \psi(x)$

$\Rightarrow$ form equation in each region I, II, III

region 1 : $-\frac{\hbar^2}{2m}\frac{\partial^2 \psi_1}{\partial t^2} = E_0 \psi_1 \quad -\infty < x < 0$

region 2 : $-\frac{\hbar^2}{2m}\frac{\partial^2 \psi_2}{\partial t^2} + U_0 \psi_2 = E_0 \psi_2 \quad 0 \le x \le L$

region 3 : $-\frac{\hbar^2}{2m}\frac{\partial^2 \psi_3}{\partial t^2} = E_0 \psi_3 \quad 0 < x < \infty$

continuity requires $\psi_1(0) = \psi_2(0)$

$\psi_2(L) = \psi_3(L)$

continuity of derivative requires $\left.\frac{d\psi_1}{dx}\right|_0 = \left.\frac{d\psi_2}{dx}\right|_0$

$\left.\frac{d\psi_2}{dx}\right|_L = \left.\frac{d\psi_3}{dx}\right|_L$

region 1 : no potential so free wave

$\Rightarrow$ incident wave moving to right

so $\psi_{in} = I e^{ikx}$

amplitude of $\psi_{in} = I$

probability of measuring incoming wave is $I^* I$, or $|I|^2$

$\Rightarrow$ reflected wave moving to left

so $\psi_r = Re^{-ikx}$

amplitude of $\psi_r = R$

probability of measuring reflected wave is R^*R or $|R|^2$

total wave function in region 1:

$$\psi_1(x) = \psi_{in}(x) + \psi_r(x)$$
$$= Ie^{ikx} + Re^{-ikx}$$

region 3: also no potential and only a transmitted wave

$$\psi_3(x) = \psi_t(x) = Te^{ikx}$$

amplitude of ψ_3 is T

probability of measuring transmitted wave is T^*T or $|T|^2$

we don't know I, R, T but if we did then we can calculate probability of incident wave getting through barrier to be

$$P_{trans} = \frac{|T|^2 \leftarrow \text{transmission}}{|I|^2 \leftarrow \text{incident}}$$

how can something get through the barrier? have to examine region 2!

note: in region 1, 2, 3, $E_0 = KE = \frac{p^2}{2m}$

use $p = \hbar k$ to write $E_0 = \frac{\hbar^2 k^2}{2m}$

solve for $k = \frac{2m\sqrt{E}}{\hbar}$

region 2: Schrodinger equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_2(x) + U_0\psi_2(x) = E_0\psi_2(x)$$

rewrite: $-\frac{\hbar^2}{2m}\frac{\partial^2\psi_2}{\partial x^2} = (E_0 - U_0)\psi_2(x)$

$E_0 < U_0$ so $E_0 - U_0 < 0$

then $\frac{\partial^2\psi_2}{\partial x^2} = \frac{-2m}{\hbar^2}(E_0 - U_0)\psi_2(x)$

$$= \underbrace{\frac{2m}{\hbar^2}(U_0 - E_0)}_{\text{positive quantity} \equiv \beta^2}\psi_2(x)$$

so $\frac{\partial^2\psi_2(x)}{\partial x^2} = \beta^2\psi_2(x)$

general solution is $\psi_2(x) = Ae^{-\beta x} + Be^{\beta x}$

$$\beta = \frac{\sqrt{2m(U_0 - E_0)}}{\hbar}$$

as $U_0 \to \infty$, barrier gets higher and $\beta \to \infty$

so $e^{-\beta x} \to 0$

but $e^{\beta x} \to \infty$ so B must be very small but non zero!

now apply continuity conditions

continuity of ψ at boundaries

region 1: $\psi_1 = Ie^{ikx} + Re^{-ikx}$

2: $\psi_2 = Ae^{-\beta x} + Be^{+\beta x}$

3: $\psi_3 = Te^{ikx}$

region 1/2 boundary: $\psi_1(0) = \psi_2(0)$

so $I + R = A + B$

2/3 " $\psi_2(L) = \psi_3(L)$

$Ae^{-\beta L} + Be^{\beta L} = Te^{ikL}$

continuity of derivatives:

region 1/2 boundary: $\left.\frac{d\psi_1}{dx}\right|_0 = \left.\frac{d\psi_2}{dx}\right|_0$

$\left.\frac{d\psi_1}{dx}\right|_0 = ikIe^{ikx}\Big|_0 - ikRe^{-ikx}\Big|_0$

$= ikI - ikR$

$\left.\frac{d\psi_2}{dx}\right|_0 = -\beta Ae^{-\beta x}\Big|_0 + \beta Be^{\beta x}\Big|_L$

so $ik(I-R) = -\beta A + \beta B$

region 2/3 boundary: $\left.\frac{d\psi_2}{dx}\right|_L = \left.\frac{d\psi_3}{dx}\right|_L$

so $-\beta A e^{-\beta L} + \beta B e^{\beta L} = ikTe^{ikL}$

So we have 4 equations

$I + R = A + B$ continuity at 1/2

$Ae^{-\beta L} + Be^{\beta L} = Te^{ikL}$ " " 2/3

$ik(I-R) = -\beta A + \beta B$ " of $\frac{d\psi}{dt}$ at 1/2

$-\beta A e^{-\beta L} + \beta B e^{\beta L} = ikTe^{ikL}$ " " " " 2/3

what we want is the transmission probability

$$P(\text{transmission}) = \frac{|T|^2}{|I|^2} = \left|\frac{T}{I}\right|^2$$

so take the 4 equations and divide by I to get

$$1 + \frac{R}{I} = \frac{A}{I} + \frac{B}{I}$$

$$\frac{A}{I}e^{-\beta L} + \frac{B}{I}e^{\beta L} = \frac{T}{I}e^{ikL}$$

$$ik\left(1 - \frac{R}{I}\right) = -\beta\frac{A}{I} + \beta\frac{B}{I}$$

$$-\beta\frac{A}{I}e^{-\beta L} + \beta\frac{B}{I}e^{\beta L} = ik\frac{T}{I}e^{ikL}$$

4 equations, 4 unknowns: $\frac{A}{I}, \frac{B}{I}, \frac{R}{I}, \frac{T}{I}$

solve for T/I after a lot of algebra!

$$\frac{T}{I} = \frac{e^{-kL}}{\cosh(\beta L) + i\left(\frac{\gamma}{2}\right)\sinh(\beta L)}$$

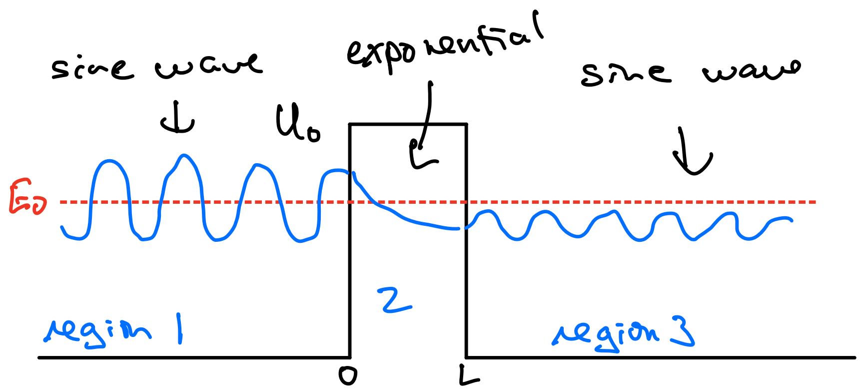
where $\cosh(\beta L) = \dfrac{e^{\beta L}+e^{-\beta L}}{2}$, $\sinh(\beta L) = \dfrac{e^{\beta L}-e^{-\beta L}}{2}$ } hyperbolic functions

$$\gamma = \frac{\beta}{k} - \frac{k}{\beta}$$

then $P(\text{transmission}) = \left|\dfrac{T}{I}\right|^2$

$$= \frac{1}{\cosh^2(\beta L) + \left(\frac{\gamma}{2}\right)^2 \sinh^2(\beta L)}$$

very strong dependence on barrier width L



This is "quantum tunneling" and is basis of solid state transistor technology!